D'Alembert's paradox

From Wikipedia, the free encyclopedia
(Redirected from D'Alembert's Paradox)

D'Alembert's paradox states that an inviscid (non-viscous), incompressible flow produces no drag on an object surrounded by such fluid, and it does not produce any lift. It is named after Jean le Rond d'Alembert. In other words, the net force which a moving inviscid uncompressible fluid exerts on the body is zero.

Discussion

If a fluid is viscous, then its flow cannot be irrotational:

viscous → ¬ irrotational (1)

This is not true for some trivial cases, like fluid at rest or in uniform motion, but becomes evident with the simplest non-trivial viscous flows, like the Poiseuille flow. Conversely, if flow is irrotational, then the fluid is inviscid:

irrotational → ¬ viscous. (from proposition (1), by Modus tollens)

But if flow is solenoidal (rotational), then must it also be viscous? (i.e. is the converse of proposition (1) also true?) Solenoidal has been defined as incompressible, not as rotational. However, solenoidal has a definite connotation of being rotational.

if ($\nabla \cdot \mathbf{v} = 0$ → viscous) then ($\nabla \cdot \mathbf{v} = 0$ → $\nabla \times \mathbf{v} \neq 0$).

What if $\nabla \cdot \mathbf{v} = 0$ (i.e. divergence is zero) and $\nabla \times \mathbf{v} = 0$ (i.e. curl is zero)? Then the fluid is Laplacian. An object moving through a Laplacian fluid which is at rest (except locally for its displacement by and around the object) suffers no drag from the fluid. This is a paradox: real fluids produce drag. For example: air is known to produce drag, otherwise parachutes would be useless. If a gas produces drag, how could a liquid not produce drag? (Since liquids are denser than gases. On the other hand, gases are compressible, and D'Alembert's paradox applies only to incompressible fluids.)

Perhaps the answer to this paradox is that Laplacian fluids do not really exist in nature: they are a mathematical abstraction. This would imply that

$$\nabla \cdot \mathbf{v} = 0 \rightarrow \nabla \times \mathbf{v} \neq 0. \qquad (2)$$

This means that solenoidal fields, which have been defined as having zero divergence (solenoidal ↔ incompressible), also rotational. But the word "solenoidal" connotes "rotational", so proposition (2) appears to make sense as a resolution of the paradox.

Proposition (2) means that:

solenoidal → rotational.

But st. (2) was derived by assuming that:

solenoidal → viscous

or

rotational → viscous.

If this is true then

viscous ↔ rotational (warning: this might not be true).

Therefore

inviscid ↔ irrotational (warning: this might not be true),

and we are back to square one: the definition of D'Alembert's Paradox: *Laplacian flow produces no drag*.

Examples and further discussion

Rotational means non-zero curl. Proposition (1) above stated that

viscous → rotational,

but does

rotational → viscous?

Example One: let the velocity field **v** be defined by

$$\mathbf{v} = -y\mathbf{i} + x\mathbf{j} \qquad (3)$$

This is an infinitely large uniform vortex (uniform : it moves like a solid). But it cannot be real (see ontology): it would have infinite kinetic energy. Curl is constant and non-zero. But is it viscous? The answer appears to be no: because the vortex moves uniformly as if it were a solid, not like a liquid. Solids are not viscous. Or if solids are viscous, then they are infinitely so: viscosity does not apply to solids. Also, equation (3) could describe the result of a passive transformation: the fluid is actually not moving (**v** = 0), but the observer at the origin is rotating with constant angular speed. Then the fluid has no chance of behaving in a viscous way.

Example Two: Now imagine a finite, cylindrical uniform vortex:

$$\mathbf{v} = -y\mathbf{i} + x\mathbf{j} \qquad \text{if } \sqrt{x^2+y^2} \le 1,$$

$$\mathbf{v} = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2+y^2} \qquad \text{if } \sqrt{x^2+y^2} \ge 1.$$

The vorticity (curl) is constant inside the vortex and zero outside it: a discontinuity of the curl at the boundary of the vortex (but the velocity field itself is continuous and differentiable).

Outside the vortex the fluid is still moving in circular streamlines, but the vorticity is zero. How can this be so? Because the vorticity of the circular motion is cancelled out exactly by shearing: be the deceleration of the fluid with distance: this shearing strain by itself is viscous, but it is cancelled out by the circular motion: irrotational, therefore non-viscous. Is viscosity acting here? Apparently not, even though the vortex is rotational, at least not mathematically, but perhaps it is acting physically.

Example Three: Imagine a jet stream, fluid moving uniformly inside an imaginary tube, like a laser beam. There is no physical boundary between the jet stream and the surrounding motionless fluid. The vorticity at the boundary is infinite, everywhere else it is zero: vorticity is discontinuous (and the velocity field is discontinuous, and perhaps not everywhere differentiable). The problem is lack of viscosity. Viscosity would smooth out the vorticity (curl). Viscosity is a type of friction which dissipates energy. It is a shearing force.

Example Four: Change the last example so that the jet stream moves around in a ring (a torus). Let the fluid inside the ring move as if it were a solid. Then the vorticity throughout the ring is constant and non-zero, but the vorticity outside the ring is constant and zero. The velocity field is discontinuous and not everywhere differentiable, because vorticity at the boundaries of the ring is infinite.

If the ring were a solid and the surroundings were also solid, then the infinite vorticity would be an indicator of the place where friction between the solids would occur. With solids, friction is localized at the interfaces between different solids. With fluids, viscosity is spread out throughout the fluid and tends to smooth out discontinuities in the velocity field.

Viscosity is a shearing force: $(F/A)/d$. It is a reaction, not a cause. The cause is vorticity which is due to shear: velocities which are parallel and adjacent but unequal in magnitude. Regions of high vorticity are hot spots which viscosity would tend to reduce and diffuse.

Inviscid fluids are a mathematical idealization: all fluids should have some viscosity;

rotational → viscous

then is trivially true for physical fluids. Then non-trivial physical flows are both viscous and rotational, not irrotational.

A liquid is, for practical purposes, incompressible (indeed, that is how liquids retain volume when they change containers. Cf. Piaget's test for concrete operational stage of cognitive development). If it were also non-viscous, and therefore irrotational, then the liquid could not be stirred with a spoon to form a vortex: the spoon could not drag the liquid, because the liquid produces no drag on the spoon. Mathematically, this is due to the liquid being irrotational and therefore Laplacian. Physically, this is due to the fluid being non-viscous: it is unable to attach itself frictionally to the spoon.

External link

- Article on d'Alembert's paradox by A. Dorsey (http://www.phys.virginia.edu/classes/311/notes/fluids1/fluids11/node19.html)

Categories: Physical paradoxes | Fluid mechanics